Section 8.6 and 8.7 Quadratic Functions and Their Graphs

$f(x) = a(x-h)^2 + k$ is the graphing form of a quadratic function.

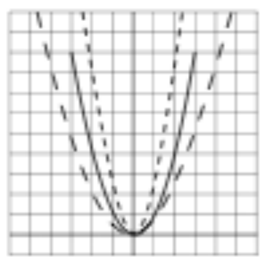
This is to quadratic functions what $y = mx + b$ is to straight lines.

If $a > 0$, the function opens up and thus has a minimum value.
If $a < 0$, the function opens down and has a maximum value.

The maximum (or minimum) value is called the "vertex" of the function.
The vertex has coordinates (h,k)

Comparing to the standard $f(x) = x^2$, a function with $0 < a < 1$ opens wider.

Comparing to the standard $f(x) = x^2$, a function with $a > 1$, opens narrower.

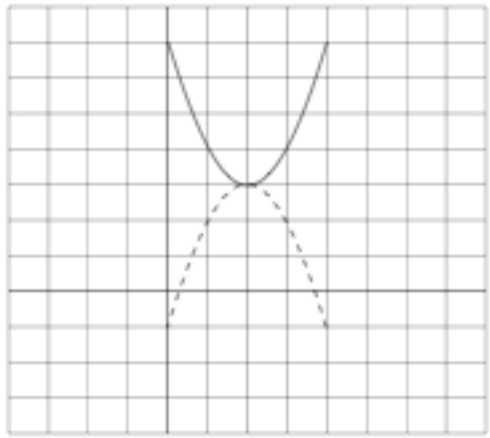


The widest graph has equation $g(x) = \frac{1}{2}x^2$.

The "middle" graph has equation $f(x) = x^2$

The narrow graph has equation $h(x) = 2x^2$

The "axis of the function" is the function's vertical center. This axis has the equation $x = h$. Both of these functions have $x = 2$ as their axis:



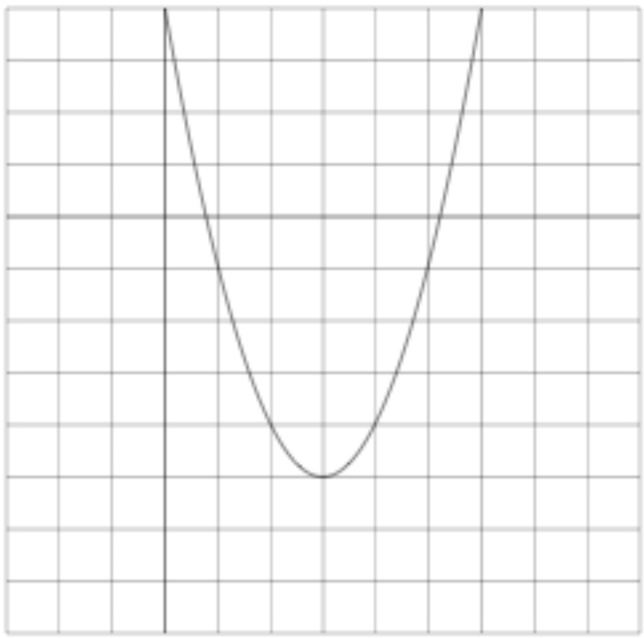
The top function is: $f(x) = (x-2)^2 + 3$.
The bottom function is: $g(x) = -(x-2)^2 + 3$.
Notice that the bottom function open downward since the "a" number is negative one.

We use "Completing the Square" to put a standard quadratic function into graphing form:

$g(x) = x^2 - 6x + 4$

$g(x) = x^2 - 6x + 9 + 4 - 9$ We divided 6 by 2 and squared. We added AND subtracted the 9

$\mathrm{g}(\mathrm{x}) = (x-3)^2 - 5$



Notice that the vertex is at (3, – 5).

Where does it cross the x-axis? We need to solve:

$$x^2 - 6x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \begin{array}{l} a = 1 \\ b = -6 \\ c = 4 \end{array}$$

We end up with $3 \pm \sqrt{5}$. About how much is that?

Well, $2 < \sqrt{5} < 3$ because $\sqrt{4} < \sqrt{5} < \sqrt{9}$ so we have one value 3–2…. Which is between zero and one and the other point would be 3+2…. Which is between 5 and 6. Look at the graph. Do these values look correct? Basically, we can draw rather accurate graphs without the need for a calculator.

One last example of completing the square with a function.

You should notice that this version is slightly different than the version we used in the beginning of the chapter. We cannot put the constant number on the other side of the equation. We just "move it over" to make room for the new constant remembering to subtract the new constant *on the same line*.

$$f(x) = 3x^2 + 18x + 13$$
$$f(x) = 3\left(x^2 + 6x + \quad\right) + 13$$
$$f(x) = 3\left(x^2 + 6x + 9\right) + 13 - 27$$ Why do we have a – 27?

Because we have a 3 on the outside of the parentheses and a 9 inside. We added $3 \cdot 9 = 27$!

$$f(x) = 3(x+3)^2 - 14$$

The vertex has coordinates: $(-3,-14)$ and the axis of the function is $x = -3$.

$$3x^2 + 18x + 13 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \begin{aligned} a &= 3 \\ b &= 18 \\ c &= 13 \end{aligned}$$

$$x = \frac{-18 \pm \sqrt{324 - 156}}{6}$$

$$x = \frac{-18 \pm \sqrt{168}}{6} \qquad 168 = 8 \cdot 21 = 2^3 \cdot 3 \cdot 7 = 2\sqrt{42} \qquad \text{and } 6 < \sqrt{42} < 7$$

$$x = \frac{2\left(-9 \pm \sqrt{42}\right)}{6}$$

$$x = \frac{-9 \pm \sqrt{42}}{3}$$

So we have – 9 – 6…..divided by 3 giving us around – 5…. Between – 6 and – 5. Also we have – 9 + 6….. giving us around – 2….. Between – 2 and – 3 This is divided by 3 so we have a number between zero and negative one.

Finally notice that $f(0) = 13$.

We can sketch a rather accurate picture by locating the vertex (– 3, – 14), plotting the coordinate (0, 13) and the two *x* intercepts and drawing a smooth curve through all 4 points.

Let's try one more to be sure we are following all the steps, in the proper order.

Find the vertex and all axis intercepts for

$f(x) = -2x^2 + 10x - 7$

$f(x) = -2\left(x^2 - 5x + \quad\right) - 7$ Notice the negative in front of the 5.

$f(x) = -2\left(x^2 - 5x + \left(\frac{5}{2}\right)^2\right) - 7 + 2\left(\frac{25}{4}\right)$

$f(x) = -2\left(x - \frac{5}{2}\right)^2 - \frac{14}{2} + \frac{25}{2}$

$f(x) = -2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}$

We will use the quadratic formula to find the coordinates of the *x*-intercepts.

$-2x^2 + 10x - 7 \overset{set}{=} 0$ I do not like the leading coefficient negative so we multiply by (-1)

$2x^2 - 10x + 7 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad a = 2$, $b = -10$, $c = 7$

$x = \dfrac{10 \pm \sqrt{100 - 56}}{4}$

$x = \dfrac{10 \pm \sqrt{4 \cdot 11}}{4}$

$x = \dfrac{10 \pm 2\sqrt{11}}{4}$

$x = \dfrac{2\left(5 \pm \sqrt{11}\right)}{4}$

$x = \dfrac{5 \pm \sqrt{11}}{2}$

$3 < \sqrt{11} < 4$ so $5 + 3.xxx$ is about 8.xxx divided by 2 is a bit more than 4.

$5 - 3.xxx$ is about 1.xxx divided by 2 is between zero and one.

The *x*-intercepts are between 0 and 1 and between 4 and 5.

It is obvious that $f(0) = -7$ so the *y*-intercept is $(0, -7)$.

The vertex, since that we have completed the square is $\left(\frac{5}{2}, \frac{11}{2}\right)$.

Finally, consider another method finding the vertex:

$f(x) = -2x^2 + 10x - 7$

$$a = -2$$
$$b = 10$$
$$c = -7$$
$$\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$$
$$\left(\frac{-10}{-4}, -7 - \frac{100}{-8}\right)$$
$$\left(\frac{-10}{-4}, -7 + \frac{100}{8}\right)$$
$$\left(\frac{5}{2}, \frac{-56+100}{8}\right)$$
$$\left(\frac{5}{2}, \frac{44}{8}\right)$$
$$\left(\frac{5}{2}, \frac{11}{2}\right)$$

As before, we found our vertex to be $\left(\frac{5}{2}, \frac{11}{2}\right)$.